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# Zeros of ultraspherical polynomials and the Hilbert–Klein formulas

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**Abstract**

The orthogonality of the ultraspherical polynomials $C_n^\lambda(z)$ for $\lambda > -\frac{1}{2}$ ensures that all of their zeros are in the interval $(-1,1)$. In a previous paper (Driver and Duren, Indag. Math. 11 (2000) 43–51), we have shown that when $\lambda < 1-n$, all of the zeros lie on the imaginary axis. Our purpose is now to describe the trajectories of the zeros of $C_n^\lambda(z)$ as $\lambda$ decreases from $-\frac{1}{2}$ to $1-n$. In particular, the pattern of migration from the interval $(-1,1)$ to the imaginary axis serves to confirm and "explain" the classical formulas of Hilbert and Klein for the number of zeros of $C_n^\lambda(z)$ lying in each of the real intervals $(-\infty,-1)$, $(-1,1)$, and $(1,\infty)$. © 2001 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

The *ultraspherical polynomials* $C_n^\lambda(z)$, also known as *Gegenbauer polynomials*, are the special case of Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ in which $\alpha = \beta = \lambda - \frac{1}{2}$. They are defined by the generating relation

$$(1-2zr+r^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(z)\, r^n.$$

For $\lambda > -\frac{1}{2}$ these polynomials $C_n^\lambda(z)$ are orthogonal over the interval $(-1,1)$ with respect to the weight function $(1-x^2)^{\lambda-1/2}$ and therefore have all of their zeros in that interval.

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There is a natural connection between ultraspherical polynomials and hypergeometric polynomials. The *hypergeometric function* $F(a,b;c;z)$ is defined by

$$F(a,b;c;z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k\,(b)_k}{k!(c)_k} z^k, \quad |z| < 1,$$

where

$$(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) = \Gamma(\alpha+k)/\Gamma(\alpha)$$

is Pochhammer's symbol. If $\alpha = -n$, a negative integer, the series terminates and is a polynomial of degree $n$. There are several formulas linking ultraspherical and hypergeometric polynomials, but the one most suitable for our purpose is

$$F(-n,b;2b;z) = \frac{n!2^{-2n}z^n}{(b+\frac{1}{2})_n} C_n^{\lambda}\left(1 - \frac{2}{z}\right), \quad \lambda = \tfrac{1}{2} - b - n. \tag{1}$$

(See [5], p. 465, no. 152; or [6], p. 64, Eq. (4.22.1).)

Strictly speaking, the polynomial $F(-n,b;2b;z)$ is not defined when $b$ is a negative integer. If $b = -m$ for some positive integer $m < n$, it is customary to interpret $F(-n,-m;-2m;z)$ as a polynomial of degree $m$. However, it is more appropriate here to adopt the convention

$$F(-n,-m;-2m;z) = \lim_{b \to -m} F(-n,b;2b;z),$$

which ensures that the zeros of $F$ vary continuously with $b$. Thus, for example,

$$F(-4,-1;-2;z) = 1 - 2z + z^3 - \tfrac{1}{2}z^4$$

instead of $1 - 2z$.

We have shown in a previous paper [1] that for $b > -\frac{1}{2}$ the zeros of $F(-n,b;2b;z)$ are all on the circle $|z-1| = 1$. But according to the above formula (1), the zeros of $C_n^{\lambda}(w)$ correspond to those of $F(-n,b;2b;z)$ by the simple relation $w = 1 - 2/z$, where $\lambda = \frac{1}{2} - b - n$. This is a linear fractional transformation which sends the circle $|z-1| = 1$ to the imaginary axis $\mathrm{Re}\{w\} = 0$, while preserving the real axis. It therefore follows that for $\lambda < 1 - n$, all zeros of $C_n^{\lambda}(w)$ lie on the imaginary axis. Moreover, for arbitrary real $\lambda$ the zeros of $C_n^{\lambda}(z)$ are symmetric with respect to both the real and imaginary axis, since we found in [1] that the zeros of $F(-n,b;2b;z)$ are all symmetric with respect to the circle $|z-1| = 1$. The symmetry of the zeros of $C_n^{\lambda}(z)$ with respect to the real axis is obvious because all coefficients are real, and the symmetry with respect to the imaginary axis then follows more easily from the formula

$$C_n^{\lambda}(-z) = (-1)^n\, C_n^{\lambda}(z),$$

which is immediate from the generating relation.

In another paper [2], we have given a detailed description of the trajectories of zeros of $F(-n,b;2b;z)$ as $b$ descends from $-\frac{1}{2}$ to $1 - n$. We found that the zeros undergo a specific series of gyrations about the singular points 0, 1, and $\infty$ while migrating from the circle $|z-1| = 1$ to the real interval $(1,\infty)$. In view of the relation (1), we have only to apply the mapping $w = 1 - 2/z$ to deduce corresponding information about the trajectories of zeros of the ultraspherical polynomials $C_n^{\lambda}(w)$ as $\lambda$ decreases from $-\frac{1}{2}$ to $1-n$. We shall therefore begin (Section 2) by recalling the precise pattern of migration of the zeros of $F(-n,b;2b;z)$, as determined analytically in [2].

## 2. Zeros of $F(-n, b; 2b; z)$

For $b > -\frac{1}{2}$, all zeros of the polynomial $F(-n,b;2b;z)$ lie on the circle $|z-1|=1$, with $z \neq 0$, and are symmetric with respect to the real axis. Note that $F(-n,b;2b;2)=0$ for all $b$ if $n$ is an odd integer. As $b$ decreases to $-\frac{1}{2}$, two zeros move symmetrically along the circle and meet at the origin. As $b$ descends below $-\frac{1}{2}$, these two zeros leave the origin in the directions $\pm 1$, hence along the real axis. For reasons of symmetry, the zeros must remain on the real axis, one on the interval $(0,1)$, the other at the symmetric point on the negative real axis, as $b$ decreases from $-\frac{1}{2}$ to $-\frac{3}{2}$. As $b$ reaches $-\frac{3}{2}$, these two zeros return along the real axis to meet two more zeros coming from the circle in a 4-way collision at the origin. As $b$ decreases below $-\frac{3}{2}$, these 4 zeros leave the origin in the directions $(-1)^{1/4}$, hence in the 4 directions $\pm 1 \pm i$. As $b$ descends to $-\frac{5}{2}$, these 4 zeros return to the origin to meet two more zeros from the circle in a 6-way collision, approaching in the directions $(-1)^{1/6}$ and leaving in directions $(+1)^{1/6}$. As $b$ reaches $-\frac{7}{2}$ , the 6 zeros return to the origin and meet two more zeros from the circle in an 8-way collision in the directions $(+1)^{1/8}$, departing in the directions $(-1)^{1/8}$. The process continues in this manner until $b$ descends to the value $\frac{1}{2}-[n/2]$, when the last two zeros leave the circle (with the exception of the fixed zero at $z=2$ if $n$ is odd). At any stage of the process equally many zeros are in each of the four "quadrants" formed by the circle $|z-1|=1$ and the real axis, at relative positions dictated by symmetry requirements. Once inside one of these "quadrants", a zero must stay inside until $b$ descends to the next negative half-integer, when all free zeros return to the origin. If $\frac{1}{2}-j-1 < b < \frac{1}{2}-j$, where $j=1,2,\ldots,[n/2]$, there are exactly $2j$ zeros detached from the circle, and $j$ of these zeros are inside the circle, the other $j$ at symmetric points outside. If $j=2k$ is even, then there are $k$ zeros in each of the four "quadrants". If $j=2k+1$ is odd, then there are $k$ zeros in each of the four "quadrants" and the remaining 2 zeros are at symmetric points on the real axis.

As $b$ decreases below the value $\frac{1}{2}-[n/2]$, the second stage of the process begins, the relocation of zeros on the real axis, eventually all on the interval $(1,\infty)$. If $n=2m$ is even, the last pair of zeros leaves the circle when $b=\frac{1}{2}-m$, and the next event occurs as $b$ descends to $-m$. Then the $m$ zeros inside the circle all collide at the point $z=1$, approaching in the directions $-(-1)^{1/m}$, while their symmetric points outside the circle "collide" at infinity. Note that if $m$ is odd, so that one zero is on the interval $(0,1)$ when $b$ falls below $\frac{1}{2}-m$, this formula dictates that it will approach the point 1 from the left (with direction $+1$) as $b \to -m$, whereas no zero can approach 1 along the real axis from the right (with direction $-1$). As $b$ descends below $-m$, the $m$ zeros leave the point 1 in directions $(+1)^{1/m}$; in particular, one of the zeros will slip onto the interval $(1,2)$ and its symmetric point will "emerge from infinity" onto the interval $(2,\infty)$. The $m$ zeros stay inside the circle, with one in the interval $(1,2)$, until $b$ descends to the value $-m-1$, when $m-1$ of the zeros return to the point 1, colliding there along directions $-(-1)^{1/(m-1)}$. Since the direction of approach $-1$ is not allowed, the zero in the interval $(1,2)$ must remain there; in fact, it tends monotonically to 2 as $b$ decreases to $-\infty$. As $b$ drops below $-m-1$, the other $m-1$ zeros leave the point 1 in directions $(+1)^{1/(m-1)}$, and in particular another zero passes into the interval $(1,2)$, while its symmetric companion emerges from infinity into the interval $(2,\infty)$. As $b$ descends further to $-m-2$, one of these zeros stays in the interval $(1,2)$ and the other $m-2$ inside the circle return to the point 1, colliding in the directions $-(-1)^{1/(m-2)}$. The process continues in this manner until $b$ descends to the value $-m-(m-1)=1-n$, when a single zero comes to the point 1 along the real axis from the left, emerging into the interval $(1,2)$, while its symmetric zero emerges from infinity into the interval $(2,\infty)$. At this stage all zeros are real and in the interval $(1,\infty)$, and they remain there as $b$ decreases further, actually converging to 2 as $b \to -\infty$.

Note that all zeros are real for the first time when $b$ descends below the value $2-n$, although two of them are then situated in the interval $(-\infty,1)$. This event is the "mirror–image" of the first event that occurs when $-\frac{3}{2} < b < -\frac{1}{2}$, when all zeros are on the circle except for one real zero on the interval $(0,1)$ and its symmetric point on the interval $(-\infty,0)$. This presupposes that $n \geqslant 4$.

If $n=2m+1$ is odd, the second half of the process is very similar, but the last two zeros (except for the fixed zero at 2) leave the circle when $b=\frac{1}{2}-m$ and then $m$ of them collide at the point 1 not when $b=-m$ but when $b=-m-1$. Then $m-1$ of the zeros return to the center of the circle as $b$ descends further to $-m-2$, and so forth until $b$ descends to the value $-2m=1-n$, when the last zero inside the circle hits the point 1 and moves into the interval $(1,2)$. Note that once again all zeros are real when $b<2-n$, and all are real and greater than 1 when $b<1-n$. Here we assume again that $n \geqslant 4$.

In the paper [2] the process of migration is illustrated for $n=8$ with some figures generated by *Mathematica*.

## 3. Zeros of ultraspherical polynomials

We have observed from the relation (1) that the zeros of the ultraspherical polynomials $C_n^{\lambda}(w)$ correspond to those of the hypergeometric polynomials $F(-n,b;2b;z)$ via the linear fractional mapping $w=1-2/z$. The correspondence of key points on the real axis is given by the table:

| | | | | |
|---|---|---|---|---|
| $z$ | 0 | 1 | 2 | $\infty$ |
| $w$ | $\infty$ | $-1$ | 0 | 1 |

In particular, the interval $1<z<\infty$ corresponds to the interval $-1<w<1$, the interval $-\infty < z < 0$ to $1<w<\infty$, and the interval $0<z<1$ to $-\infty<w<-1$. The circle $|z-1|=1$ corresponds to the imaginary axis $Re\{w\}=0$.

As the parameter $\lambda$ decreases from $-\frac{1}{2}$ to $1-n$, we see that $b=\frac{1}{2}-\lambda-n$ increases from $1-n$ to $-\frac{1}{2}$. Thus, the zeros of $C_n^{\lambda}(w)$ emerge from the interval $(-1,1)$ and make their way to the imaginary axis in trajectories that are images under the mapping $w=1-2/z$ of reversed trajectories of the zeros of $F(-n,b;2b;z)$. More specifically, as $\lambda$ descends below $-\frac{1}{2}$, two zeros emerge onto the real axis at opposite ends of the interval $(-1,1)$, returning to meet another pair of zeros as $\lambda$ reaches $-\frac{3}{2}$. Then two zeros depart from each end of the interval $(-1,1)$ into the upper and lower half-planes in the vertical directions $\pm i$, in paths symmetric with respect to both real and imaginary axis, performing loops within each of the four quadrants and returning to the same endpoints to meet another pair of zeros at equal angles $2\pi/3$ as $\lambda$ descends to $-\frac{5}{2}$. Three zeros then leave each endpoint of the interval $(-1,1)$ at equal angles $2\pi/3$, one along the real axis at each end, returning to the same endpoints to meet yet another pair of zeros at equal angles $\pi/2$ as $\lambda$ descends to $-\frac{7}{2}$. These oscillations continue until $\lambda$ reaches $\frac{1}{2}-[n/2]$, when the last pair of zeros leaves the interval $(-1,1)$, except for the fixed zero at the origin if $n$ is odd. At any stage of the process, equally many zeros are situated in each quadrant of the plane. If

$$\tfrac{1}{2}-j-1<\lambda<\tfrac{1}{2}-j, \quad j=1,2,\ldots,[n/2],$$

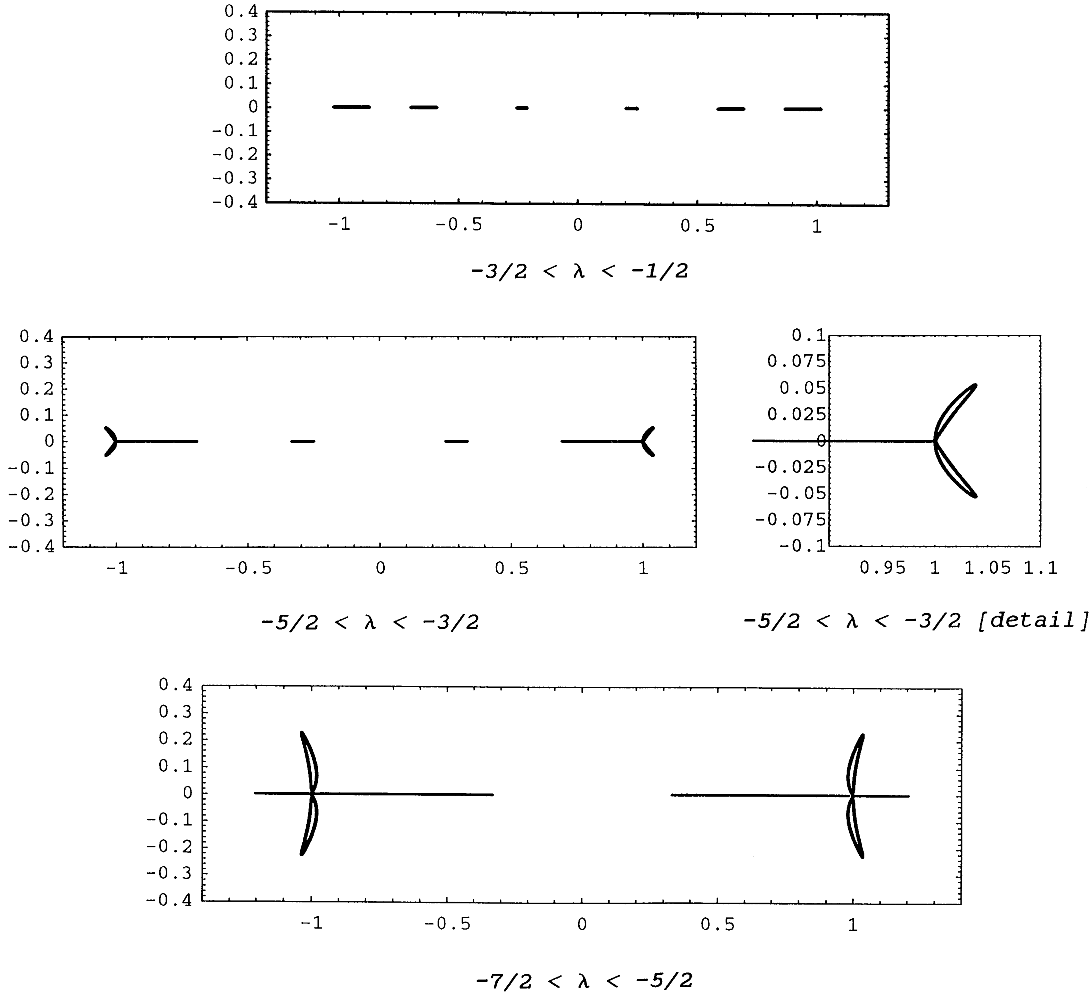


Fig. 1. Zeros of $C_8(z)$ for $-7/2 < \lambda < -1/2$.

there are exactly $2j$ zeros outside the interval $(-1,1)$, and $j$ of these zeros are in the right half-plane $\mathrm{Re}\{w\} > 0$, the other $j$ in the left half-plane. If $j = 2k$ is even, then exactly $k$ of these zeros are in each quadrant of the plane (not on the real or imaginary axis). If $j = 2k+1$ is odd, then $k$ zeros are in each of the four quadrants while the remaining 2 are on the real axis, one in the interval $(1,\infty)$, the other in $(-\infty,-1)$.

The zeros then all move to infinity, and as $\lambda$ decreases below $-[(n+1)/2]$, two of the zeros emerge onto the imaginary axis, while the others perform loops and return to infinity as $\lambda$ descends to $-1-[(n+1)/2]$. Two more zeros then pass onto the imaginary axis and the others again traverse loops, returning to infinity when $\lambda$ reaches $-2-[(n+1)/2]$. The oscillations continue in this manner until $\lambda$ falls below $1-n$, when finally all zeros are on the imaginary axis (including, of course, the fixed zero at the origin if $n$ is odd). At any stage of the process, there are equally many zeros in

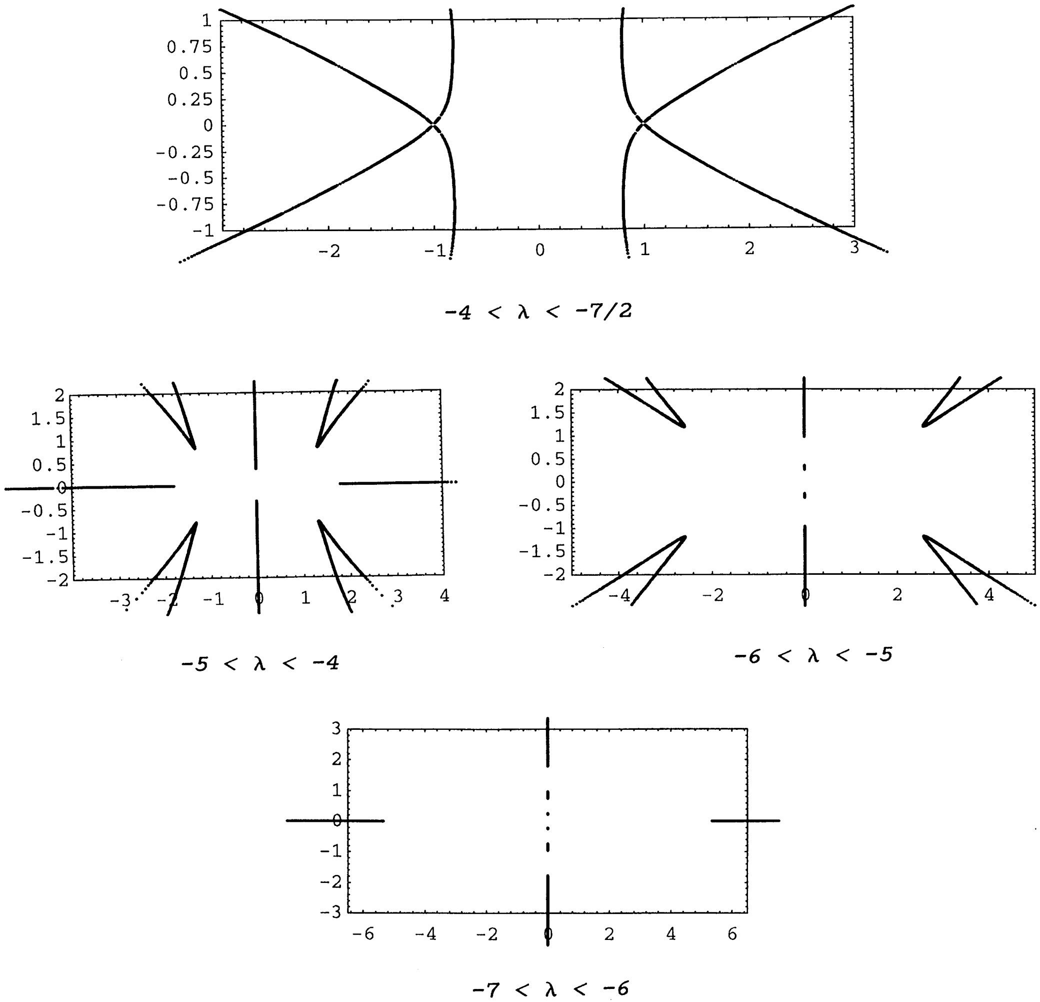


Fig. 2. Zeros of $C_8(z)$ for $-7 < \lambda < -7/2$.

each of the four quadrants of the plane, excluding the real and imaginary axes. If

$$j-n<\lambda<j-n+1, \quad j=1,2,\ldots,[n/2],$$

then $2j$ of the detached zeros have not yet reached the imaginary axis. If $j=2k$ is even, then $k$ of the free zeros are situated in each quadrant, and none are on the real axis. If $j=2k+1$ is odd, then $k$ of the zeros are in each quadrant and the remaining 2 zeros are on the real axis, one in the interval $(1,\infty)$, the other in $(-\infty,-1)$. In the latter case two of the "loops" from infinity degenerate into linear trajectories along the real axis.

Figs. 1 and 2 display *Mathematica* graphics that illustrate the process for $n=8$. Fig. 1 shows the trajectories of the 8 zeros during the first stage of the process. As $\lambda$ decreases from $-\frac{1}{2}$ to $-\frac{3}{2}$, two zeros move out horizontally from opposite ends of the interval $(-1,1)$, returning along the real axis to the same endpoints to meet another pair of zeros. As $\lambda$ decreases from $-\frac{3}{2}$ to $-\frac{5}{2}$, two zeros leave the endpoints vertically in opposite directions, making loops (shown also in detail) and returning to the same endpoints to meet yet another pair of zeros which have moved horizontally within the interval $(-1,1)$. At each endpoint, the 3 zeros return at equal angles $2\pi/3$ as $\lambda$ descends to $-\frac{5}{2}$. Then as $\lambda$ decreases from $-\frac{5}{2}$ to $-\frac{7}{2}$, one of these zeros slips out of the interval $(-1,1)$ along the real axis, while the other two leave again at equal angles $2\pi/3$ and perform loops in the complex plane. As $\lambda$ descends to $-\frac{7}{2}$, the 3 zeros detached from each end return to their respective endpoints $\pm 1$ to meet the last pair of zeros, which have slid through the interval $(-1,1)$ in preparation for departure. The 4 zeros at each endpoint meet at right angles.

Fig. 2 shows the second stage of the migration process. As $\lambda$ decreases from $-\frac{7}{2}$ to $-4$, all 8 zeros move off to infinity. As $\lambda$ decreases from $-4$ to $-5$, one pair of zeros moves symmetrically along the imaginary axis from infinity toward the origin, while the other 6 perform symmetric "loops" in from infinity, returning to infinity as $\lambda$ reaches $-5$. Note that one pair of loops actually degenerates to trajectories along the real axis, in from infinity and back. Then as $\lambda$ decreases from $-5$ to $-6$, a second pair of zeros joins the first pair on the imaginary axis, sliding symmetrically from infinity toward the origin, while the other 4 zeros perform loops in each quadrant, returning to infinity as $\lambda$ approaches $-6$. As $\lambda$ decreases from $-6$ to $-7$, a third pair of zeros joins the first two pairs on the imaginary axis, sliding from infinity toward the origin, while the remaining pair perform degenerate loops from infinity along the real axis, returning to infinity as $\lambda$ descends to $-7$. Finally, as $\lambda$ descends below $-7$, all 8 zeros are on the imaginary axis, tending symmetrically to the origin as $\lambda \to -\infty$.

## 4. The Hilbert–Klein formulas

The classical formulas of Hilbert [3] and Klein [4], also known to Stieltjes, give the numbers $N_1, N_2$, and $N_3$ of zeros of an ultraspherical polynomial $C_n^{\lambda}(z)$ that lie in the respective intervals $(-1,1), (-\infty,-1)$, and $(1,\infty)$ of the real axis. These formulas were actually given more generally for Jacobi polynomials (cf. [6, p. 144]) and are expressed in terms of Klein's symbol

$$E(u) = \begin{cases} 0, & u \leqslant 0 \\ [u], & u > 0, \ u \neq 1,2,\dots \\ u-1, & u = 1,2,\dots\ . \end{cases}$$

Two quantities $X$ and $Y$ are defined by formulas which specialize in the ultraspherical case to

$$X = E(|n+\lambda| - |\lambda - \tfrac{1}{2}| + \tfrac{1}{2}),$$
$$Y = E(\tfrac{1}{2} - |n+\lambda|).$$

Then the Hilbert–Klein formulas for ultraspherical polynomials say that

$$N_1 = \begin{cases} 2[(X+1)/2], & n \text{ even}, \\ 2[X/2]+1, & n \text{ odd}; \end{cases}$$

while

$$N_2 = N_3 = \begin{cases} 2[(Y+1)/2] & \text{if} \dbinom{2n+2\lambda-1}{n}\dbinom{n+\lambda-\frac{1}{2}}{n} > 0, \\ 2[Y/2]+1 & \text{if} \dbinom{2n+2\lambda-1}{n}\dbinom{n+\lambda-\frac{1}{2}}{n} < 0. \end{cases}$$

It is to be expected that $N_2 = N_3$, since the zeros of $C_n^\lambda(z)$ are symmetric with respect to the imaginary axis. Note that $Y = 0$ for all $n$ and all $\lambda$, so that the last formula takes the simpler form

$$N_2 = N_3 = \begin{cases} 0 & \text{if} \dbinom{2n+2\lambda-1}{n}\dbinom{n+\lambda-\frac{1}{2}}{n} > 0, \\ 1 & \text{if} \dbinom{2n+2\lambda-1}{n}\dbinom{n+\lambda-\frac{1}{2}}{n} < 0. \end{cases}$$

From our foregoing description of the trajectories of the zeros of $C_n^\lambda(z)$ as $\lambda$ decreases below $-\frac{1}{2}$, we can easily predict the numbers $N_1, N_2$, and $N_3$ for each range of $\lambda$ between successive integers or half-integers. Of course, when $\lambda > -\frac{1}{2}$, we know that $N_1 = n$ and $N_2 = N_3 = 0$. For $-\frac{3}{2} < \lambda < -\frac{1}{2}$, we have seen that two zeros of $C_n^\lambda(z)$ emerge onto the real axis at opposite ends of the interval $(-1, 1)$, so that $N_1 = n-2$ and $N_2 = N_3 = 1$. When $-\frac{5}{2} < \lambda < -\frac{3}{2}$, four zeros of $C_n^\lambda(z)$ have left the real axis and moved into the complex plane, while the rest of the $n-4$ zeros remain in the interval $(-1, 1)$. Thus $N_1 = n-4$ and $N_2 = N_3 = 0$. As $\lambda$ decreases further, this process of the "peeling off" of zeros of $C_n^\lambda(z)$, two at a time from opposite ends of the interval $(-1, 1)$, continues until $\lambda$ reaches the value $\frac{1}{2} - [n/2]$, when the last pair of zeros leaves $(-1, 1)$, except for the fixed zero at the origin when $n$ is odd. To simplify the discussion, let us focus on the case where $n = 2m$ is even. Then $N_1 = 0$ for $\lambda < \frac{1}{2} - m$, while $N_2 = N_3$ oscillates between 0 and 1 in successive intervals

$$-m < \lambda < \tfrac{1}{2} - m, \quad -1-m < \lambda < -m, \quad -2-m < \lambda < -1-m,$$
$$-3-m < \lambda < -2-m, \dots,$$

starting with 0 if $m$ is even and with 1 if $m$ is odd. The process terminates when $\lambda$ descends below $1-n$, when all zeros are on the imaginary axis (and different from 0 if $n$ is even), so that $N_1 = N_2 = N_3 = 0$.

These results are confirmed by the Hilbert–Klein formulas, which say that $N_2 = N_3$ can be only 0 or 1 for $\lambda < -\frac{1}{2}$, oscillating between these two values according to the sign of the binomial product

$$\binom{2n+2\lambda-1}{n}\binom{n+\lambda-\frac{1}{2}}{n}.$$

When $\lambda$ descends below $1-n$, it is easily seen once again that the sign of the given product is always positive, so that $N_2 = N_3 = 0$ for all $\lambda < 1-n$. Another simple observation is that whenever $1-n < \lambda < \frac{1}{2} - n/2$, we have $X = E(n+2\lambda) = 0$, so that $N_1 = 0$ for $n$ even and $N_1 = 1$ for $n$ odd (the fixed zero at the origin). In all cases, the numbers predicted by our results are in agreement with the Hilbert–Klein formulas. Our analysis may be viewed as an independent derivation of those formulas in the ultraspherical case, with the extra insight of the dynamical picture that explains where the zeros go when they leave the real axis.

As a final remark, we mention that a result regarding the zeros of the Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ for $\alpha=-2\beta-2n-1$ and $\beta>-1$, obtained in a previous paper [1, Theorem 5] is also in agreement with the more general form of the Hilbert–Klein formulas, since it says that all zeros lie in the interval $(-\infty,-1)$. It would be interesting to find a general "dynamical interpretation" of the Hilbert–Klein formulas for Jacobi polynomials, through a corresponding analysis of the trajectories of zeros when one of the parameters $\alpha$ and $\beta$ descends below $-1$.

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